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A Local Effective Action for Photon–Gravity Interactions

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ABSTRACT: Quantum phenomena such as vacuum polarisation in curved spacetime induce interactions between photons and gravity with quite striking consequences, including the violation of the strong equivalence principle and the apparent prediction of 'superluminal' photon propagation. These quantum interactions can be encoded in an effective action. In this paper, we extend previous results on the effective action for QED in curved spacetime due to Barvinsky, Vilkovisky and others and present a new, local effective action valid to all orders in a derivative expansion, as required for a full analysis of the quantum theory of high-frequency photon propagation in gravitational fields.

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1. Introduction

In classical electrodynamics in curved spacetime, the interaction of the electromagnetic and gravitational fields is encoded in the Maxwell action

$$\Gamma_{(0)} \quad = \quad -\frac{1}{4}\int dx\sqrt{g}\; F_{\mu\nu}F^{\mu\nu} \tag{1.1}$$

where the field strength $F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu$ is defined with covariant derivatives. The interaction involves only the connection and is independent of the spacetime curvature. This action therefore embodies the strong equivalence principle (SEP), which states that the laws of physics should be the same in the local inertial frames at each point in spacetime and reduce to their special relativistic form at the origin of each of these frames. This is assured since the connections (though not of course the curvatures) vanish locally in an appropriate frame in a Riemannian manifold.

The picture is rather different in quantum theory. In quantum electrodynamics in curved spacetime, vacuum polarisation loops involving the electron induce interactions between photons and the background gravitational field that depend explicitly on the curvature, with the necessary scale being set by the Compton wavelength of the electron, $\lambda_c = \frac{1}{m}$. These interactions therefore effectively violate the SEP at the quantum level.

The simplest effective action which encodes some of the essential physics of vacuum polarisation was constructed explicitly by Drummond and Hathrell [1], who used standard Schwinger–de Witt proper time/heat kernel methods (see e.g. [2,3,4,5,6]) as well as an independent diagrammatic technique to find

$$\begin{aligned}\Gamma_{\rm DH} \quad = \quad \Gamma_{(0)} + \frac{1}{m^2}\int dx\sqrt{g}\;\Big(&aRF_{\mu\nu}F^{\mu\nu} + bR_{\mu\nu}F^{\mu\lambda}F^{\nu}{}_{\lambda} + cR_{\mu\nu\lambda\rho}F^{\mu\nu}F^{\lambda\rho}\\ &+ dD_\mu F^{\mu\lambda}D_\nu F^{\nu}{}_{\lambda}\Big)\end{aligned} \tag{1.2}$$

Here, a, b, c, d are perturbative coefficients of $O(\alpha)$, viz.

$$a = -\frac{1}{144}\frac{\alpha}{\pi} \qquad b = \frac{13}{360}\frac{\alpha}{\pi} \qquad c = -\frac{1}{360}\frac{\alpha}{\pi} \qquad d = -\frac{1}{30}\frac{\alpha}{\pi} \tag{1.3}$$

As well as being a one-loop result, there are two other significant limitations on $\Gamma_{\rm DH}$. First, it is at most of third order in the generalised 'curvatures' (including both the Riemann curvature and the electromagnetic field strengths). Second, it is just the lowest-order term in a derivative expansion involving higher orders in $O(\frac{D}{m})$ acting on the curvatures. If we restrict ourselves to the interesting case of terms of $O(F^2)$, which determine the photon

propagator and thus the quantum effects on the propagation of light in gravitational fields, the first restriction is equivalent to working to first order in $O(\frac{R}{m^2})$, or $O(\frac{\lambda_c^2}{L^2})$ where L is a typical curvature scale, i.e. weak background gravitational fields. In terms of photon propagation, the second condition implies a restriction to low frequencies, i.e. lowest order in $O(\frac{\lambda_c}{\lambda})$ where λ is the photon wavelength.

By far the most dramatic consequence of eq.(1.2) is its prediction of superluminal photon velocities. As explained in ref.[1] (see refs.[7,8,9,10,11,12] for a selection of subsequent work), the new SEP-violating interactions induce curvature-dependent shifts in the light cones, which become

$$k^2 \;-\; \frac{1}{m^2}(2b+4c)R_{\mu\lambda}k^\mu k^\lambda \;+\; \frac{8c}{m^2}C_{\mu\nu\lambda\rho}k^\mu k^\lambda a^\nu a^\rho \;=\; 0 \tag{1.4}$$

where k^μ is the photon momentum and a^μ is its polarisation. Note that we have written eq.(1.4) in a convenient form involving the Weyl tensor $C_{\mu\nu\lambda\rho}$. For suitably chosen trajectories in certain curved spacetimes, this condition remarkably permits k^2 to be spacelike.

Although photon propagation is the most striking application of the effective action, it is clearly important much more generally in encoding the often unintuitive effects of gravity on quantum fields. In this paper, we present an extension of the Drummond-Hathrell action in which we relax the restriction to lowest order in the derivative expansion, while retaining the weak gravitational field approximation. The application of this improved effective action to the photon propagation problem and the nature of dispersion in gravitational fields is considered separately in ref.[12]; here we discuss the technical issues arising in the construction of the effective action itself.

At one-loop order, the QED effective action is given (in Euclidean space) by

$$\Gamma = \Gamma_{(0)} + \ln \det S(x,x') \tag{1.5}$$

where $\Gamma_{(0)}$ is the free Maxwell action and $S(x,x')$ is the Green function of the Dirac operator in the background gravitational field, i.e.

$$\big(i\not{D} - m\big)S(x,x') = \frac{1}{\sqrt{g}}\delta(x,x') \tag{1.6}$$

In fact it is more convenient to work with the differential operator corresponding to the scalar Green function $G(x,x')$ defined by

$$S(x,x') = \big(i\not{D} + m\big)G(x,x') \tag{1.7}$$

so that

$$\Big(D^2 + ie\sigma^{\mu\nu}F_{\mu\nu} - \frac{1}{4}R + m^2\Big)G(x,x') \;=\; -\frac{1}{\sqrt{g}}\delta(x,x') \tag{1.8}$$

Then we evaluate Γ from the heat kernel, or proper time, representation

$$\Gamma \;=\; \Gamma_{(0)} \;-\; \frac{1}{2}\int_0^\infty \frac{ds}{s}\; e^{-m^2 s}\; \mathrm{Tr}\mathcal{G}(x,x';s) \tag{1.9}$$

where

$$\mathcal{D}\mathcal{G}(x,x';s) = \frac{\partial}{\partial s}\mathcal{G}(x,x';s) \tag{1.10}$$

with $\mathcal{G}(x,x';0) = \frac{1}{\sqrt{g}}\delta(x,x')$. Here, $\mathcal{D}$ is the differential operator in eq.(1.8) at $m=0$.

Our principal result is summarised in the following expression for the one-loop QED effective action, to first order in $O(\frac{R}{m^2})$:

$$\begin{aligned}\Gamma = \int dx\sqrt{g}\Big[-\frac{1}{4}Z\; F_{\mu\nu}F^{\mu\nu} \;+\; &\frac{1}{m^2}\Big(D_\mu F^{\mu\lambda}\;\overrightarrow{G_0}\;D_\nu F^\nu{}_\lambda \\ &+\;\overrightarrow{G_1}\;RF_{\mu\nu}F^{\mu\nu} \;+\; \overrightarrow{G_2}\;R_{\mu\nu}F^{\mu\lambda}F^\nu{}_\lambda \;+\; \overrightarrow{G_3}\;R_{\mu\nu\lambda\rho}F^{\mu\nu}F^{\lambda\rho}\Big) \\ +\;&\frac{1}{m^4}\Big(\overrightarrow{G_4}\;RD_\mu F^{\mu\lambda}D_\nu F^\nu{}_\lambda \\ &+\;\overrightarrow{G_5}\;R_{\mu\nu}D_\lambda F^{\lambda\mu}D_\rho F^{\rho\nu} \;+\; \overrightarrow{G_6}\;R_{\mu\nu}D^\mu F^{\lambda\rho}D^\nu F_{\lambda\rho} \\ &+\;\overrightarrow{G_7}\;R_{\mu\nu}D^\mu D^\nu F^{\lambda\rho}F_{\lambda\rho} \;+\; \overrightarrow{G_8}\;R_{\mu\nu}D^\mu D^\lambda F_{\lambda\rho}F^{\rho\nu} \\ &+\;\overrightarrow{G_9}\;R_{\mu\nu\lambda\rho}D_\sigma F^{\sigma\rho}D^\lambda F^{\mu\nu}\;\Big)\;\Big]\end{aligned} \tag{1.11}$$

The $\overrightarrow{G_n}$ $(n\geq 1)$ are form factor functions of three operators:

$$\overrightarrow{G_n} \equiv G_n\Big(-\frac{D^2_{(1)}}{m^2}, -\frac{D^2_{(2)}}{m^2}, -\frac{D^2_{(3)}}{m^2}\Big) \tag{1.12}$$

where the first entry ($D^2_{(1)}$) acts on the first following term (the curvature), etc. $\overrightarrow{G_0}$ is similarly defined as a single variable function. The explicit expressions for these form factors are presented later in the paper.

Although the entire effective action is a non-local object, for theories such as QED with a massive electron, it should permit a local expansion in inverse powers of m. A crucial feature of the above form of the effective action is that it is indeed manifestly local, in the sense that the form factors $\overrightarrow{G_n}$ have an expansion in positive powers of the $D^2_{(i)}$. This depends on making the particular choice of basis operators above.

A very general effective action calculation, which encompasses the more specialised results we need, has been given some time ago by Barvinsky, Gusev, Zhytnikov and Vilkovisky (BGZV) in ref.[13], building on methods developed earlier by two of the authors

[14,15,16]. Closely related results have also been obtained by Avramidi [17]. The present paper is essentially a translation and specialisation of the BGZV result to the $O(RFF)$ terms in the QED effective action. However, the BGZV result is presented in an apparently non-local form and the translation into a manifestly local effective action through an appropriate choice of basis operators involves a number of subtle manoeuvres. Together with the complexity of the manipulations on the form factors, we feel this justifies our independent presentation, in which we explain the delicate technical points in the translation and quote explicit results for the form factors in the *local* effective action (1.11).

In the next section, we introduce the BGZV effective action and show how it is recast in local form and reduced to the $O(RFF)$ action we are looking for. Then in section 3, we reconsider the $O(F^2)$ terms and rewrite them in a convenient form consistent with the Drummond-Hathrell action. Section 4 contains a summary of our final result, with explicit algebraic expressions for the form factors collected in an appendix. A numerical analysis of the form factors is given in section 5. The paper is presented so that the final results for the local effective action and the form factors are entirely self-contained in section 4 and the appendix.

2. The BGZV Effective Action

Barvinsky *et al.* evaluate the heat kernel $\mathcal{G}(x,x';s)$ (denoted by K in refs.[13,15]) corresponding to the generic second-order elliptic differential operator

$$H = \Box\hat{\mathbf{1}} + \hat{P} - \frac{1}{6}R\hat{\mathbf{1}}, \qquad \Box \equiv g^{\mu\nu}D_\mu D_\nu \tag{2.1}$$

acting on small fluctuations $\delta\phi^A$ of an arbitrary set of fields $\phi^A(x)$. The hat notation denotes matrices acting on the vector space of the $\delta\phi^A$, i.e. $\hat{P} = P^A{}_B$, $\quad \hat{\mathbf{1}} = \delta^A{}_B$, etc. and the matrix trace is denoted tr. The Euclidean metric $g_{\mu\nu}$ describes the background spacetime, with gravitational Ricci and Riemann curvatures $R_{\mu\nu}$ and $R_{\mu\nu\lambda\rho}$ respectively. A further 'curvature' is defined by the commutators of the covariant derivatives acting on the fields $\delta\phi^A$:

$$\big(D_\mu D_\nu - D_\nu D_\mu\big)\delta\phi^A = \mathcal{R}^A{}_{B\mu\nu}\delta\phi^B \tag{2.2}$$

and we denote $\hat{\mathcal{R}}_{\mu\nu} = \mathcal{R}^A{}_{B\mu\nu}$. The 'potential' P is an arbitrary matrix. BGZV introduce the collective notation $\mathfrak{R}$ for the full set of generalised curvatures:

$$\mathfrak{R} \;\; = \;\; \big\{R_{\mu\nu\lambda\rho},\; \hat{\mathcal{R}}_{\mu\nu},\; \hat{P}\big\} \tag{2.3}$$

and calculate the heat kernel $\mathcal{G}$ up to $O(\mathfrak{R}^3)$.

The first step in extracting the QED effective action from the BGZV results is to identify the curvatures $\mathfrak{R}$ by comparing the differential operator $\mathcal{D}$ of eqs.(1.8),(1.10) with the standard form H. In this case, the field ϕ^A is the electron spinor field $\psi_\alpha(x)$, so we identify A, B as Dirac spinor indices α, β and identify

$$\hat{P} = ie\sigma^{\mu\nu}F_{\mu\nu} - \frac{1}{12}R\hat{\mathbf{1}} \tag{2.4}$$

The (gauge and gravitational) covariant derivative acting on spinors is

$$D_\mu = \partial_\mu + \frac{1}{2}\sigma^{ab}\omega_{ab\mu} + ieA_\mu \tag{2.5}$$

where ${\omega^a}_{b\mu}$ is the spin connection

$${\omega^a}_{b\mu} = {e_b}^\nu\big(\partial_\mu {e^a}_\nu - \Gamma_{\mu\nu}^\lambda {e^a}_\lambda\big) \tag{2.6}$$

with ${e^a}_\mu$ the vierbein and $\sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b]$. Evaluating the commutator, we find

$$\big[D_\mu, D_\nu\big] = ieF_{\mu\nu}\hat{\mathbf{1}} + \frac{1}{2}\sigma^{ab}R_{ab\mu\nu} \tag{2.7}$$

and so, comparing with eq.(2.2), we identify

$$\hat{\mathcal{R}}_{\mu\nu} = ieF_{\mu\nu}\hat{\mathbf{1}} + \frac{1}{2}\sigma^{\lambda\rho}R_{\mu\nu\lambda\rho} \tag{2.8}$$

Having established this identification between the BGZV generalised curvatures and the QED field strength and gravitational curvature tensors, we can now summarise their result for Tr $\mathcal{G}(x,x';s)$. Here, Tr denotes the full functional trace, i.e. Tr $\mathcal{G}(x,x';s) = \int dx\sqrt{g}$ tr $\mathcal{G}(x,x';s)\big|_{x'=x}$. The result is [13]:

$$\begin{aligned}
\mathrm{Tr}\ \mathcal{G} \quad = \quad & \frac{1}{(4\pi)^2}\frac{1}{s^2}\int dx\sqrt{g}\ \ \mathrm{tr}\ \Bigg[\hat{\mathbf{1}} + s\hat{P} \\
& + s^2\sum_{i=1}^{5} f_i(-s\Box_{(2)})\ \mathfrak{R}_1\mathfrak{R}_2(i) \\
& + s^3\sum_{i=1}^{11} F_i(-s\Box_{(1)}, -s\Box_{(2)}, -s\Box_{(3)})\ \mathfrak{R}_1\mathfrak{R}_2\mathfrak{R}_3(i) \\
& + s^4\sum_{i=12}^{25} F_i(-s\Box_{(1)}, -s\Box_{(2)}, -s\Box_{(3)})\ \mathfrak{R}_1\mathfrak{R}_2\mathfrak{R}_3(i) \\
& + s^5\sum_{i=26}^{28} F_i(-s\Box_{(1)}, -s\Box_{(2)}, -s\Box_{(3)})\ \mathfrak{R}_1\mathfrak{R}_2\mathfrak{R}_3(i) \\
& + s^6\ F_{29}(-s\Box_{(1)}, -s\Box_{(2)}, -s\Box_{(3)})\ \mathfrak{R}_1\mathfrak{R}_2\mathfrak{R}_3(29) \quad + \quad O(\mathfrak{R}^4)\ \Bigg]
\end{aligned} \tag{2.9}$$

Here, $\mathfrak{R}_1\mathfrak{R}_2(i)$ denote independent basis operators formed from terms of second order in the generalised curvatures (2.3). Similarly for the independent third order terms $\mathfrak{R}_1\mathfrak{R}_2\mathfrak{R}_3(i)$. Notice that these appear with different powers of s, determined by their dimension. The form factors f_i and F_i are functions of the operators $\Box_{(1)}$, etc. with the suffix indicating which curvature term is acted upon. Thus, for example, since $\mathfrak{R}_1\mathfrak{R}_2(5) = \hat{\mathcal{R}}_{1\mu\nu}\hat{\mathcal{R}}_2{}^{\mu\nu}$, the corresponding contribution to Tr $\mathcal{G}$ is

$$\frac{1}{(4\pi)^2}\int dx\sqrt{g}\;\hat{\mathcal{R}}_{1\mu\nu}f_5(-s\Box)\hat{\mathcal{R}}_2{}^{\mu\nu} \tag{2.10}$$

We will return to the detailed expressions for the form factors later. First, we consider the basis of 5 $O(\mathfrak{R}^2)$ and 29 $O(\mathfrak{R}^3)$ operators and, with the identifications in eqs.(2.4) and (2.8), pick out those from the full list which produce the terms of $O(F^2)$ and $O(RFF)$ relevant for our QED effective action. The remaining terms involving purely gravitational curvatures of $O(R^2)$ and $O(R^3)$ are neglected. In BGZV notation, the relevant basis operators are as follows:

$$\begin{aligned}
\mathrm{tr}\;\mathfrak{R}_1\mathfrak{R}_2(4) &= \mathrm{tr}\;\hat{P}_1\hat{P}_2 = \frac{1}{2}e^2F_{\mu\nu}F^{\mu\nu}\;\mathrm{tr}\hat{\mathbf{1}} \;+\; O(R^2)\\
\mathrm{tr}\;\mathfrak{R}_1\mathfrak{R}_2(5) &= \mathrm{tr}\;\hat{\mathcal{R}}_{1\mu\nu}\hat{\mathcal{R}}_2{}^{\mu\nu} = -e^2F_{\mu\nu}F^{\mu\nu}\;\mathrm{tr}\hat{\mathbf{1}} \;+\; O(R^2)\\
\mathrm{tr}\;\mathfrak{R}_1\mathfrak{R}_2\mathfrak{R}_3(1) &= \mathrm{tr}\;\hat{P}_1\hat{P}_2\hat{P}_3 = -\frac{1}{24}e^2\Big(RF_{\mu\nu}F^{\mu\nu}+F_{\mu\nu}RF^{\mu\nu}+F_{\mu\nu}F^{\mu\nu}R\Big)\;\mathrm{tr}\hat{\mathbf{1}} \;+\; O(R^3)\\
\mathrm{tr}\;\mathfrak{R}_1\mathfrak{R}_2\mathfrak{R}_3(3) &= \mathrm{tr}\;\hat{\mathcal{R}}_{1\mu\nu}\hat{\mathcal{R}}_2^{\mu\nu}\hat{P}_3 = \Big(\frac{1}{12}e^2F_{\mu\nu}F^{\mu\nu}R+\frac{1}{4}e^2R_{\mu\nu\lambda\rho}F^{\mu\nu}F^{\lambda\rho}\\
&\qquad +\frac{1}{4}e^2F^{\mu\nu}R_{\mu\nu\lambda\rho}F^{\lambda\rho}\Big)\;\mathrm{tr}\hat{\mathbf{1}} \;+\; O(R^3)\\
\mathrm{tr}\;\mathfrak{R}_1\mathfrak{R}_2\mathfrak{R}_3(6) &= \mathrm{tr}\;\hat{P}_1\hat{P}_2R_3 = \frac{1}{2}e^2F_{\mu\nu}F^{\mu\nu}R\;\mathrm{tr}\hat{\mathbf{1}} \;+\; O(R^3)\\
\mathrm{tr}\;\mathfrak{R}_1\mathfrak{R}_2\mathfrak{R}_3(7) &= \mathrm{tr}\;R_1\hat{\mathcal{R}}_{2\mu\nu}\hat{\mathcal{R}}_3^{\mu\nu} = -e^2RF_{\mu\nu}F^{\mu\nu}\;\mathrm{tr}\hat{\mathbf{1}} \;+\; O(R^3)\\
\mathrm{tr}\;\mathfrak{R}_1\mathfrak{R}_2\mathfrak{R}_3(8) &= \mathrm{tr}\;R_{1\mu\nu}\hat{\mathcal{R}}_2^{\mu\lambda}\hat{\mathcal{R}}_{3\,\lambda}^{\nu} = -e^2R_{\mu\nu}F^{\mu\lambda}F^{\nu}{}_{\lambda}\;\mathrm{tr}\hat{\mathbf{1}} \;+\; O(R^3)\\
\mathrm{tr}\;\mathfrak{R}_1\mathfrak{R}_2\mathfrak{R}_3(14) &= \mathrm{tr}\;D_\mu\hat{\mathcal{R}}_1^{\mu\lambda}D_\nu\hat{\mathcal{R}}_{2\,\lambda}^{\nu}\hat{P}_3 = \Big(\frac{1}{12}e^2D_\mu F^{\mu\lambda}D_\nu F^{\nu}{}_{\lambda}R+\frac{1}{4}e^2D_\mu R_{\rho\sigma}{}^{\mu\lambda}D_\nu F^{\nu}{}_{\lambda}F^{\rho\sigma}\\
&\qquad +\frac{1}{4}e^2D_\mu F^{\mu\lambda}D_\nu R_{\rho\sigma}{}^{\nu}{}_{\lambda}F^{\rho\sigma}\Big)\;\mathrm{tr}\hat{\mathbf{1}} \;+\; O(R^3)\\
\mathrm{tr}\;\mathfrak{R}_1\mathfrak{R}_2\mathfrak{R}_3(17) &= \mathrm{tr}\;R_{1\mu\nu}D^\mu D^\nu\hat{P}_2\hat{P}_3 = \frac{1}{2}e^2R_{\mu\nu}D^\mu D^\nu F^{\lambda\rho}F_{\lambda\rho}\;\mathrm{tr}\hat{\mathbf{1}} \;+\; O(R^3)\\
\mathrm{tr}\;\mathfrak{R}_1\mathfrak{R}_2\mathfrak{R}_3(18) &= \mathrm{tr}\;R_{1\mu\nu}D_\lambda\hat{\mathcal{R}}_2^{\lambda\mu}D_\rho\hat{\mathcal{R}}_3^{\rho\nu} = -e^2R_{\mu\nu}D_\lambda F^{\lambda\mu}D_\rho F^{\rho\nu}\;\mathrm{tr}\hat{\mathbf{1}} \;+\; O(R^3)\\
\mathrm{tr}\;\mathfrak{R}_1\mathfrak{R}_2\mathfrak{R}_3(19) &= \mathrm{tr}\;R_{1\mu\nu}D^\mu\hat{\mathcal{R}}_2^{\lambda\rho}D^\nu\hat{\mathcal{R}}_{3\lambda\rho} = -e^2R_{\mu\nu}D^\mu F^{\lambda\rho}D^\nu F_{\lambda\rho}\;\mathrm{tr}\hat{\mathbf{1}} \;+\; O(R^3)\\
\mathrm{tr}\;\mathfrak{R}_1\mathfrak{R}_2\mathfrak{R}_3(20) &= \mathrm{tr}\;R_1D_\mu\hat{\mathcal{R}}_2^{\mu\lambda}D_\nu\hat{\mathcal{R}}_3{}^{\nu}{}_{\lambda} = -e^2RD_\mu F^{\mu\lambda}D_\nu F^{\nu}{}_{\lambda}\;\mathrm{tr}\hat{\mathbf{1}} \;+\; O(R^3)\\
\mathrm{tr}\;\mathfrak{R}_1\mathfrak{R}_2\mathfrak{R}_3(21) &= \mathrm{tr}\;R_{1\mu\nu}D^\mu D^\lambda\hat{\mathcal{R}}_{2\lambda\rho}\hat{\mathcal{R}}_3^{\rho\nu} = -e^2R_{\mu\nu}D^\mu D^\lambda F_{\lambda\rho}F^{\rho\nu}\;\mathrm{tr}\hat{\mathbf{1}} \;+\; O(R^3)
\end{aligned} \tag{2.11}$$

Notice that for clarity we have omitted the subscripts 1,2,3 on the terms in the right-hand column, but must remember that the order is still significant since they are to be acted on by the form factors. $\hat{\mathbf{1}}$ is the unit Dirac matrix, so in four dimensions $\mathrm{tr}\hat{\mathbf{1}} = 4$.

Collecting these terms and substituting into eq.(2.9) then gives the complete $O(F^2)$ and $O(RFF)$ terms in the BGZV effective action. However, as it stands this is expressed in *non-local* form, whereas we require a manifestly local action. As explained in ref.[15], the key is to reintroduce the Riemann tensor into the basis of operators. (Notice that none of the above set of terms in the form tr $\mathfrak{R}_1\mathfrak{R}_2\mathfrak{R}_3$ involves the uncontracted Riemann tensor $R_{\mu\nu\lambda\rho}$, which only enters in the right-hand column when we expand $\hat{\mathcal{R}}_{\mu\nu}$ and $\hat{P}$ according to eqs.(2.4),(2.8). This is a feature of the covariant perturbation theory construction developed by Barvinsky and Vilkovisky.) We therefore introduce the further operator

$$\mathrm{tr}\ R_{1\mu\nu\lambda\rho}\hat{\mathcal{R}}_2^{\mu\nu}\hat{\mathcal{R}}_3^{\lambda\rho} \quad = \quad -e^2 R_{\mu\nu\lambda\rho}F^{\mu\nu}F^{\lambda\rho}\ \mathrm{tr}\hat{\mathbf{1}} \quad + \quad O(R^3) \tag{2.12}$$

The Riemann tensor admits the following non-local re-expression in terms of the Ricci tensor, up to $O(R^2)$:

$$R_{\mu\nu\lambda\rho} \quad = \quad \frac{1}{\Box}\Big(D_\mu D_\lambda R_{\nu\rho} - D_\nu D_\lambda R_{\mu\rho} - D_\mu D_\rho R_{\nu\lambda} + D_\nu D_\rho R_{\mu\lambda}\Big) \ + \ O(R^2) \tag{2.13}$$

This is proved [15] by differentiating the Bianchi identity

$$R_{\mu\nu[\lambda\rho;\sigma]} \quad = \quad 0 \tag{2.14}$$

to obtain an equation for $\Box R_{\mu\nu\lambda\rho}$, which is solved iteratively in powers of curvature using the Green function $1/\Box$ to obtain (2.13).

We can then rewrite the four BGZV basis operators which turn out to have non-local form factors as:

$$\begin{aligned}
&\int dx\sqrt{g}\ \mathrm{tr}\ \Big[sF_8\ R_{\mu\nu}\hat{\mathcal{R}}^{\mu\lambda}\hat{\mathcal{R}}^{\nu}{}_{\lambda} \quad + \quad s^2F_{18}\ R_{\mu\nu}D_\lambda\hat{\mathcal{R}}^{\lambda\mu}D_\rho\hat{\mathcal{R}}^{\rho\nu} \\
&\qquad + \ s^2F_{19}\ R_{\mu\nu}D^\mu\hat{\mathcal{R}}^{\lambda\rho}D^\nu\hat{\mathcal{R}}_{\lambda\rho} \quad + \quad s^2F_{21}\ R_{\mu\nu}D^\mu D^\lambda\hat{\mathcal{R}}_{\lambda\rho}\hat{\mathcal{R}}^{\rho\nu}\Big] \\
= \quad &\int dx\sqrt{g}\ \mathrm{tr}\ \Big[s\tilde{F}_8\ R_{\mu\nu}\hat{\mathcal{R}}^{\mu\lambda}\hat{\mathcal{R}}^{\nu}{}_{\lambda} \quad + \quad s^2\tilde{F}_{18}\ R_{\mu\nu}D_\lambda\hat{\mathcal{R}}^{\lambda\mu}D_\rho\hat{\mathcal{R}}^{\rho\nu} \\
&\qquad + \ s^2\tilde{F}_{19}\ R_{\mu\nu}D^\mu\hat{\mathcal{R}}^{\lambda\rho}D^\nu\hat{\mathcal{R}}_{\lambda\rho} \quad + \quad s^2\tilde{F}_{21}\ R_{\mu\nu}D^\mu D^\lambda\hat{\mathcal{R}}_{\lambda\rho}\hat{\mathcal{R}}^{\rho\nu} \\
&\qquad + \ s\tilde{F}_0\ R_{\mu\nu\lambda\rho}\hat{\mathcal{R}}^{\mu\nu}\hat{\mathcal{R}}^{\lambda\rho}\Big]
\end{aligned} \tag{2.15}$$

with the following relation between the form factors:

$$
\begin{aligned}
F_8(x_1,x_2,x_3) &= \tilde{F}_8(x_1,x_2,x_3) + 2\Big(1+\frac{x_2+x_3}{x_1}\Big)\tilde{F}_0(x_1,x_2,x_3) \\
F_{18}(x_1,x_2,x_3) &= \tilde{F}_{18}(x_1,x_2,x_3) - \frac{4}{x_1}\tilde{F}_0(x_1,x_2,x_3) \\
F_{19}(x_1,x_2,x_3) &= \tilde{F}_{19}(x_1,x_2,x_3) + \frac{2}{x_1}\tilde{F}_0(x_1,x_2,x_3) \\
F_{21}(x_1,x_2,x_3) &= \tilde{F}_{21}(x_1,x_2,x_3) - \frac{8}{x_1}\tilde{F}_0(x_1,x_2,x_3)
\end{aligned}
\tag{2.16}
$$

where we have used the notation $x_i = -\Box_{(i)}$. (Notice that BGZV [13] use $\xi_i = -s\Box_{(i)}$ in discussing the form factors.)

The new form factor $\tilde{F}_0(x_1,x_2,x_3)$ is defined so as to remove the singular terms in $F_{8,18,19,21}$ leaving the $\tilde{F}_{8,18,19,21}$ form factors regular as $x_1,x_2,x_3 \to 0$. Of course this under-determines $\tilde{F}_0$, but any ambiguity only amounts to a reshuffling of contributions amongst the various terms in the final expression for the effective action.

In the local basis, we therefore find that all the $O(F^2)$ and $O(RFF)$ terms in the heat kernel are contained in the following BGZV terms (again suppressing the arguments of the form factors, i.e. $f_i \equiv f_i(-s\Box_{(2)})$ and $F_i \equiv F_i(-s\Box_{(1)}, -s\Box_{(2)}, -s\Box_{(3)})$):

$$
\begin{aligned}
\int dx\sqrt{g}\ \mathrm{tr}\Big[\ & f_4\ \hat{P}_1\hat{P}_2 \ + \ f_5\ \hat{\mathcal{R}}_{1\mu\nu}\hat{\mathcal{R}}_2{}^{\mu\nu} \\
&+ sF_1\ \hat{P}_1\hat{P}_2\hat{P}_3 \ + \ sF_3\ \hat{\mathcal{R}}_{1\mu\nu}\hat{\mathcal{R}}_2^{\mu\nu}\hat{P}_3 \ + \ sF_6\ \hat{P}_1\hat{P}_2R_3 \ + \ sF_7\ R_1\hat{\mathcal{R}}_{2\mu\nu}\hat{\mathcal{R}}_3^{\mu\nu} \\
&+ s^2F_{14}\ D_\mu\hat{\mathcal{R}}_1^{\mu\lambda}D_\nu\hat{\mathcal{R}}_{2\lambda}^{\nu}\hat{P}_3 \ + \ s^2F_{17}\ R_{1\mu\nu}D^\mu D^\nu\hat{P}_2\hat{P}_3 \ + \ s^2F_{20}\ R_1D_\mu\hat{\mathcal{R}}_2^{\mu\lambda}D_\nu\hat{\mathcal{R}}_3{}^{\nu}{}_{\lambda} \\
&+ s\tilde{F}_8\ R_{1\mu\nu}\hat{\mathcal{R}}_2^{\mu\lambda}\hat{\mathcal{R}}_{3\lambda}^{\nu} \ + \ s^2\tilde{F}_{18}\ R_{1\mu\nu}D_\lambda\hat{\mathcal{R}}_2^{\lambda\mu}D_\rho\hat{\mathcal{R}}_3^{\rho\nu} \\
&+ s^2\tilde{F}_{19}\ R_{1\mu\nu}D^\mu\hat{\mathcal{R}}_2^{\lambda\rho}D^\nu\hat{\mathcal{R}}_{3\lambda\rho} \ + \ s^2\tilde{F}_{21}\ R_{1\mu\nu}D^\mu D^\lambda\hat{\mathcal{R}}_{2\lambda\rho}\hat{\mathcal{R}}_3^{\rho\nu} \\
&+ s\tilde{F}_0\ R_{\mu\nu\lambda\rho}\hat{\mathcal{R}}^{\mu\nu}\hat{\mathcal{R}}^{\lambda\rho}\ \Big]
\end{aligned}
\tag{2.17}
$$

We can now re-express this explicitly in terms of the $O(F^2)$ and $O(RFF)$ basis operators of eq.(1.11) using the set of relations (2.11). We also need to use the following identity, which is proved by integrating by parts and using the Bianchi identity for $F_{\mu\nu}$:

$$
\begin{aligned}
&\int dx\sqrt{g}\ D^\lambda R_{\mu\nu\lambda\rho}D_\sigma F^{\sigma\rho}F^{\mu\nu} \\
&= -\int dx\sqrt{g}\ \Big(\ R_{\mu\nu\lambda\rho}D_\sigma F^{\sigma\rho}D^\lambda F^{\mu\nu} \ + \ \frac{1}{2}R_{\mu\nu\lambda\rho}\Box F^{\mu\nu}F^{\lambda\rho} \ + \ O(R^2)\ \Big)
\end{aligned}
\tag{2.18}
$$

This gives:

$$
\begin{aligned}
\mathrm{Tr}\ \mathcal{G} \ \ = \ \ -\frac{e^2}{(4\pi)^2}\mathrm{tr}\hat{\mathbf{1}} \int dx\sqrt{g}\, \Big[\ & F_{\mu\nu}\ h_0\ F^{\mu\nu} \\
+ \ & s\left(h_1\ RF_{\mu\nu}F^{\mu\nu} \ + \ h_2\ R_{\mu\nu}F^{\mu\lambda}F^{\nu}{}_{\lambda} \ + \ h_3\ R_{\mu\nu\lambda\rho}F^{\mu\nu}F^{\lambda\rho}\right) \\
+ \ & s^2\Big(h_4\ RD_{\mu}F^{\mu\lambda}D_{\nu}F^{\nu}{}_{\lambda} \\
& \quad + \ h_5\ R_{\mu\nu}D_{\lambda}F^{\lambda\mu}D_{\rho}F^{\rho\nu} \ + \ h_6\ R_{\mu\nu}D^{\mu}F^{\lambda\rho}D^{\nu}F_{\lambda\rho} \\
& \quad + \ h_7\ R_{\mu\nu}D^{\mu}D^{\nu}F^{\lambda\rho}F_{\lambda\rho} \ + \ h_8\ R_{\mu\nu}D^{\mu}D^{\lambda}F_{\lambda\rho}F^{\rho\nu} \\
& \quad + \ h_9\ R_{\mu\nu\lambda\rho}D_{\sigma}F^{\sigma\rho}D^{\lambda}F^{\mu\nu}\ \Big)\ \Big]
\end{aligned}
\tag{2.19}
$$

where $h_0 \equiv h_0(-s\Box)$ and $h_i \equiv h_i(-s\Box_{(1)}, -s\Box_{(2)}, -s\Box_{(3)})$, $\ i \geq 1$. The form factors are related to the BGZV definitions as follows (where we assume appropriate symmetrisations on x_1, x_2, x_3):

$$
\begin{aligned}
h_0(x) \ &= \ -\frac{1}{2}f_4(x) \ + \ f_5(x) \\
h_1(x_1,x_2,x_3) \ &= \ \frac{1}{8}F_1(x_1,x_2,x_3) \ - \ \frac{1}{12}F_3(x_2,x_3,x_1) \\
& \quad - \frac{1}{2}F_6(x_2,x_3,x_1) \ + \ F_7(x_1,x_2,x_3) \\
h_2(x_1,x_2,x_3) \ &= \ \tilde{F}_8(x_1,x_2,x_3) \\
h_3(x_1,x_2,x_3) \ &= \ -\frac{1}{2}F_3(x_1,x_2,x_3) \ - \ \frac{1}{8}F_{14}(x_1,x_2,x_3)(x_2+x_3) \\
& \quad + \ \tilde{F}_0(x_1,x_2,x_3) \\
h_4(x_1,x_2,x_3) \ &= \ -\frac{1}{12}F_{14}(x_2,x_3,x_1) \ + \ F_{20}(x_1,x_2,x_3) \\
h_5(x_1,x_2,x_3) \ &= \ \tilde{F}_{18}(x_1,x_2,x_3) \\
h_6(x_1,x_2,x_3) \ &= \ \tilde{F}_{19}(x_1,x_2,x_3) \\
h_7(x_1,x_2,x_3) \ &= \ -\frac{1}{2}F_{17}(x_1,x_2,x_3) \\
h_8(x_1,x_2,x_3) \ &= \ \tilde{F}_{21}(x_1,x_2,x_3) \\
h_9(x_1,x_2,x_3) \ &= \ \frac{1}{2}F_{14}(x_1,x_2,x_3)
\end{aligned}
\tag{2.20}
$$

3. Second Order Terms

While eq.(2.19) is a perfectly satisfactory, local expression for Tr $\mathcal{G}$, it is not the most convenient form for many applications, e.g. photon propagation. Nor is it the form which permits the simplest comparison with standard Schwinger–deWitt results. The key step in transforming to this standard form is to rewrite the second-order contributions of the type $\int dx\sqrt{g}\ F_{\mu\nu}\Box F^{\mu\nu}$ as $\int dx\sqrt{g}\ D_\mu F^{\mu\lambda}D_\nu F^\nu{}_\lambda$, plus higher order terms of $O(RFF)$ which of course modify the corresponding form factors. This second choice for the $O(F^2)$ basis operator also matches the term in the Drummond-Hathrell action (1.2) and turns out to be the more useful representation.

This transformation can be made already at the level of the BGZV curvatures using the identity [15,13]:

$$\begin{aligned}\mathrm{tr}\int dx\sqrt{g}\ \hat{\mathcal{R}}_{\mu\nu}\Box\hat{\mathcal{R}}^{\mu\nu} \ &= \ \mathrm{tr}\int dx\sqrt{g}\ \Big(-2D_\mu\hat{\mathcal{R}}^{\mu\lambda}D_\nu\hat{\mathcal{R}}^\nu{}_\lambda \ + \ 4\hat{\mathcal{R}}^\mu{}_\lambda\hat{\mathcal{R}}^\lambda{}_\rho\hat{\mathcal{R}}^\rho{}_\mu \\ &\qquad + 2R_{\mu\nu}\hat{\mathcal{R}}^{\mu\lambda}\hat{\mathcal{R}}^\nu{}_\lambda \ - \ R_{\mu\nu\lambda\rho}\hat{\mathcal{R}}^{\mu\nu}\hat{\mathcal{R}}^{\lambda\rho}\Big)\end{aligned} \tag{3.1}$$

In terms of the electromagnetic field strength and gravitational curvature, this is equivalent to

$$\begin{aligned}\int dx\sqrt{g}\ F_{\mu\nu}\Box F^{\mu\nu} \ &= \ \int dx\sqrt{g}\ \Big(-2D_\mu F^{\mu\lambda}D_\nu F^\nu{}_\lambda \\ &\qquad + \ 2R_{\mu\nu}F^{\mu\lambda}F^\nu{}_\lambda \ - \ R_{\mu\nu\lambda\rho}F^{\mu\nu}F^{\lambda\rho} \ + \ O(R^2)\Big)\end{aligned} \tag{3.2}$$

The proof is straightforward, relying on the Bianchi identity for $F_{\mu\nu}$ and the cyclic symmetry of the Riemann tensor.

We now need to extend this identity to include the full form factors. The first step is to separate off the constant term in $h_0(-\Box)$ and define

$$\int dx\sqrt{g}\ F_{\mu\nu}h_0(-\Box)F^{\mu\nu} \ = \ h_0(0)\int dx\sqrt{g}\ F_{\mu\nu}F^{\mu\nu} \ + \int dx\sqrt{g}\ F_{\mu\nu}h(-\Box)\Box F^{\mu\nu} \tag{3.3}$$

The first term on the r.h.s., where $h(0) = -1/6$, eventually gives rise to a divergent contribution to the effective action which is removed by the conventional electromagnetic field renormalisation. The remaining term simplifies in two stages. First, write

$$\begin{aligned}\int dx\sqrt{g}\ F_{\mu\nu}h(-\Box)\Box F^{\mu\nu} \ &= \ \int dx\sqrt{g}\ \Big(-2D_\mu F^{\mu\lambda}h(-\Box)D_\nu F^\nu{}_\lambda \ + \ 2R_{\mu\nu}F^{\mu\lambda}h(-\Box)F^\nu{}_\lambda \\ &\qquad - R_{\mu\nu\lambda\rho}F^{\mu\nu}h(-\Box)F^{\lambda\rho} \ + \ 2D_\mu F^{\mu\lambda}\big[D_\nu, h(-\Box)\big]F^\nu{}_\lambda\Big)\end{aligned} \tag{3.4}$$

The second and third terms are already in the required form of $O(RFF)$ basis operators. Neglecting higher-order terms in the curvature simplifies the commutator, and with repeated use of the Bianchi identity for the Riemann tensor we eventually find

$$\begin{aligned}\int dx\sqrt{g}\; F_{\mu\nu}h(-\Box)\Box F^{\mu\nu} \;=\; \int dx\sqrt{g}\,\Big(&-2D_\mu F^{\mu\lambda}h(-\Box)D_\nu F^\nu{}_\lambda \\ &+2R_{\mu\nu}\big(h(-\Box)+\Box h'(-\Box)\big)F^{\mu\lambda}F^\nu\lambda \;-\; R_{\mu\nu\lambda\rho}h(-\Box)F^{\mu\nu}F^{\lambda\rho} \\ &-2R_{\mu\nu}h'(-\Box)D_\lambda F^{\lambda\mu}D_\rho F^{\rho\nu} \;+\; 4R_{\mu\nu}h'(-\Box)D^\mu D^\lambda F_{\lambda\rho}F^{\rho\nu} \\ &+2R_{\mu\nu\lambda\rho}h'(-\Box)D_\sigma F^{\sigma\rho}D^\lambda F^{\mu\nu} \;+\; O(R^2FF)\Big)\end{aligned} \tag{3.5}$$

The transformation between the alternative choices for the $O(F^2)$ basis operator therefore affects the $O(RFF)$ form factors h_2, h_3, h_5, h_8 and h_9 , in the notation of (2.19).

With this transformation, we can rewrite Tr $\mathcal{G}$ into the preferred form:

$$\begin{aligned}\mathrm{Tr}\;\mathcal{G} \;=\; -\frac{e^2}{(4\pi)^2}\mathrm{tr}\hat{\mathbf{1}}\int dx\sqrt{g}\,\Big[&-\frac{1}{6}F_{\mu\nu}F^{\mu\nu}+s\;D_\mu F^{\mu\lambda}\;g_0\;D_\nu F^\nu{}_\lambda \\ &+\;s\,\Big(g_1\;RF_{\mu\nu}F^{\mu\nu}\;+\;g_2\;R_{\mu\nu}F^{\mu\lambda}F^\nu{}_\lambda\;+\;g_3\;R_{\mu\nu\lambda\rho}F^{\mu\nu}F^{\lambda\rho}\Big) \\ &+\;s^2\,\Big(g_4\;RD_\mu F^{\mu\lambda}D_\nu F^\nu{}_\lambda \\ &\quad+\;g_5\;R_{\mu\nu}D_\lambda F^{\lambda\mu}D_\rho F^{\rho\nu}\;+\;g_6\;R_{\mu\nu}D^\mu F^{\lambda\rho}D^\nu F_{\lambda\rho} \\ &\quad+\;g_7\;R_{\mu\nu}D^\mu D^\nu F^{\lambda\rho}F_{\lambda\rho}\;+\;g_8\;R_{\mu\nu}D^\mu D^\lambda F_{\lambda\rho}F^{\rho\nu} \\ &\quad+\;g_9\;R_{\mu\nu\lambda\rho}D_\sigma F^{\sigma\rho}D^\lambda F^{\mu\nu}\Big)\Big]\end{aligned} \tag{3.6}$$

with $g_0 \equiv g_0(-s\Box)$ and $g_i \equiv g_i(-s\Box_{(1)}, -s\Box_{(2)}, -s\Box_{(3)})$, $\;i \geq 1$ as before. The new form factors are related to the initial ones by:

$$\begin{aligned}g_0(x) &= -2h(x) \;=\; \frac{2}{x}\Big(-\frac{1}{2}f_4(x)\;+\;f_5(x)\;+\;\frac{1}{6}\Big) \\ g_{1,4,6,7} &= h_{1,4,6,7} \\ g_2 &= h_2+h(x_2)+x_2h'(x_2)+h(x_3)+x_3h'(x_3) \\ g_3 &= h_3-\frac{1}{2}\big(h(x_2)+h(x_3)\big) \\ g_5 &= h_5-\big(h'(x_2)+h'(x_3)\big) \\ g_8 &= h_8-4h'(x_2) \\ g_9 &= h_9+2h'(x_2)\end{aligned} \tag{3.7}$$

where $g_i \equiv g_i(x_1, x_2, x_3)$, $\;i \geq 1$ and we have symmetrised where appropriate.

4. The Local Effective Action

Collecting these results and evaluating the effective action from

$$\Gamma \;=\; \Gamma_0 \;-\; \frac{1}{2}\int_0^\infty \frac{ds}{s}\; e^{-m^2 s}\; \mathrm{Tr}\; \mathcal{G}(x,x';s) \tag{4.1}$$

we find the renormalised expression:

$$\begin{aligned}\Gamma = \int dx\sqrt{g}\Big[-\frac{1}{4}Z\; F_{\mu\nu}F^{\mu\nu} \;&+\; \frac{1}{m^2}\Big(D_\mu F^{\mu\lambda}\; \overrightarrow{G_0}\; D_\nu F^\nu{}_\lambda \\ &\qquad + \overrightarrow{G_1}\; RF_{\mu\nu}F^{\mu\nu} \;+\; \overrightarrow{G_2}\; R_{\mu\nu}F^{\mu\lambda}F^\nu{}_\lambda \;+\; \overrightarrow{G_3}\; R_{\mu\nu\lambda\rho}F^{\mu\nu}F^{\lambda\rho}\Big) \\ &+\; \frac{1}{m^4}\Big(\overrightarrow{G_4}\; RD_\mu F^{\mu\lambda}D_\nu F^\nu{}_\lambda \\ &\qquad + \overrightarrow{G_5}\; R_{\mu\nu}D_\lambda F^{\lambda\mu}D_\rho F^{\rho\nu} \;+\; \overrightarrow{G_6}\; R_{\mu\nu}D^\mu F^{\lambda\rho}D^\nu F_{\lambda\rho} \\ &\qquad + \overrightarrow{G_7}\; R_{\mu\nu}D^\mu D^\nu F^{\lambda\rho}F_{\lambda\rho} \;+\; \overrightarrow{G_8}\; R_{\mu\nu}D^\mu D^\lambda F_{\lambda\rho}F^{\rho\nu} \\ &\qquad + \overrightarrow{G_9}\; R_{\mu\nu\lambda\rho}D_\sigma F^{\sigma\rho}D^\lambda F^{\mu\nu}\Big)\Big]\end{aligned} \tag{4.2}$$

where the form factors $\overrightarrow{G_n} = G_n\big(-\frac{\Box_{(1)}}{m^2}, -\frac{\Box_{(2)}}{m^2}, -\frac{\Box_{(3)}}{m^2}\big)$ are defined as proper time integrals of the g_n, viz.

$$G_n(x_1,x_2,x_3) \;=\; -\frac{1}{2}\frac{\alpha}{\pi}\int_0^\infty \frac{ds}{s}\; e^{-s}s^p\; g_n(sx_1,sx_2,sx_3) \tag{4.3}$$

where $p = 1$ for $n = 0,\ldots 3$ and $p = 2$ for $n = 4,\ldots 9$. Note that we have rescaled s here so that it is dimensionless, while now $x_i = -\Box_{(i)}/m^2$. The divergent contribution to the $F_{\mu\nu}F^{\mu\nu}$ term has been evaluated using dimensional regularisation and cancelled by the standard electromagnetic field renormalisation. The residual finite $O(\alpha)$ contribution gives $Z = 1 + \frac{1}{6}\frac{\alpha}{\pi}\ln\frac{m^2}{\bar{\mu}^2}$ for a suitable choice of RG scale $\bar{\mu}$.

The form factors are given in terms of the BGZV definitions by the following set of relations:

$$\begin{aligned} g_0(x) \;&=\; -2h(x) \;=\; \frac{2}{x}\Big(-\frac{1}{2}f_4(x) \;+\; f_5(x) \;+\; \frac{1}{6}\Big) \\ g_1(x_1,x_2,x_3) \;&=\; \frac{1}{8}F_1(x_1,x_2,x_3) \;-\; \frac{1}{12}F_3(x_2,x_3,x_1) \\ &\qquad -\frac{1}{2}F_6(x_2,x_3,x_1) \;+\; F_7(x_1,x_2,x_3) \\ g_2(x_1,x_2,x_3) \;&=\; \tilde{F}_8(x_1,x_2,x_3) \;+\; h(x_2) + x_2h'(x_2) + h(x_3) + x_3h'(x_3)\end{aligned}$$

$$
\begin{aligned}
g_3(x_1,x_2,x_3) &= -\frac{1}{2}F_3(x_1,x_2,x_3) \;-\; \frac{1}{8}F_{14}(x_1,x_2,x_3)(x_2+x_3) \\
&\quad +\; \tilde{F}_0(x_1,x_2,x_3) \;-\; \frac{1}{2}\big(h(x_2)+h(x_3)\big) \\
g_4(x_1,x_2,x_3) &= -\frac{1}{12}F_{14}(x_2,x_3,x_1) \;+\; F_{20}(x_1,x_2,x_3) \\
g_5(x_1,x_2,x_3) &= \tilde{F}_{18}(x_1,x_2,x_3) \;-\; \big(h'(x_2)+h'(x_3)\big) \\
g_6(x_1,x_2,x_3) &= \tilde{F}_{19}(x_1,x_2,x_3) \\
g_7(x_1,x_2,x_3) &= -\frac{1}{2}F_{17}(x_1,x_2,x_3) \\
g_8(x_1,x_2,x_3) &= \tilde{F}_{21}(x_1,x_2,x_3) \;-\; 4h'(x_2) \\
g_9(x_1,x_2,x_3) &= \frac{1}{2}F_{14}(x_1,x_2,x_3) \;+\; 2h'(x_2)
\end{aligned}
\tag{4.4}
$$

where $h'(x) \equiv \frac{dh}{dx}$ and the $\tilde{F}_i$ are given in terms of the BGZV F_i by eq.(2.16). Again we assume that the BGZV form factors have been appropriately symmetrised to reflect the symmetries of the basis operators on which they act.

The BGZV form factors themselves are known algebraic expressions which involve the following integrals [15,13]:

$$
f(x) \;=\; \int_0^1 d\alpha \; e^{-\alpha(1-\alpha)x} \tag{4.5}
$$

and

$$
F(x_1,x_2,x_3) \;=\; \int_{\alpha\geq 0} d^3\alpha \; \delta(1-\alpha_1-\alpha_2-\alpha_3)\; e^{-(\alpha_1\alpha_2 x_3+\alpha_2\alpha_3 x_1+\alpha_3\alpha_1 x_2)} \tag{4.6}
$$

The second-order form factors are easily simplified. Using the BGZV expressions,

$$
f_4(x) = \frac{1}{2}f(x) \qquad\qquad f_5(x) = -\frac{1}{2x}\big(f(x)-1\big) \tag{4.7}
$$

we have

$$
g_0(x) = \frac{1}{3x} + \frac{1}{x^2} - \Big(\frac{1}{2x} + \frac{1}{x^2}\Big)f(x) \tag{4.8}
$$

Using the small x expansion of $f(x)$, viz.

$$
f(x) = 1 - \frac{1}{6}x + \frac{1}{60}x^2 - \frac{1}{840}x^3 + \ldots \tag{4.9}
$$

we find

$$
g_0(x) = \frac{1}{15} - \frac{1}{140}x + \ldots \tag{4.10}
$$

The absence of inverse powers of x confirms that this is indeed a *local* form factor.

The expansions for the third-order form factors $F_i(x_1, x_2, x_3)$ are extremely complicated. They can be read off from the results given in ref.[13] and we will not quote them in full here. Instead, we focus on a special case of particular interest. In most applications, it is reasonable to assume that the derivatives of the gravitational curvatures are governed by the same scale as the curvatures themselves, i.e. if $\frac{R}{m^2} = O(\frac{\lambda_c^2}{L^2})$ then $\frac{DR}{m^3} = O(\frac{\lambda_c^3}{L^3})$. In that case, terms with derivatives of curvatures are suppressed by the same parameter governing the expansion in increasing powers of curvature, and should be neglected. We would then only be interested in the form factors with the first argument set to zero, i.e. $g_n(0, x_2, x_3)$.

We have calculated the corresponding form factors $F_i(0, x_2, x_3)$ and $\tilde{F}_i(0, x_2, x_3)$ appearing in eq.(4.4), identifying the new form factor $\tilde{F}_0(0, x_2, x_3)$ required for the transformation to the local basis. The resulting lengthy expressions are collected in the appendix.

These results must pass three crucial consistency checks. First, the transformation from the non-local basis to the local one via the introduction of $\tilde{F}_0(x_1, x_2, x_3)$ as described in eq.(2.16) must work, i.e. the $1/x_1$ singularities in each of the form factors $F_{8,18,19,21}$ must be removed by the same function $\tilde{F}_0$. Second, all the resulting form factors must be entirely local, in the sense that they admit Taylor expansions with no inverse powers of $x_{1,2,3}$. Finally, the final form factors $g_0(0)$ and $g_n(0,0,0)$, $n = 1, 2, 3$ must reproduce the Schwinger–de Witt coefficients in the Drummond-Hathrell effective action. In algebraic terms, each of these tests is highly non-trivial, but as can be verified from the explicit results in the appendix, each is indeed satisfied.

The $x_{1,2,3} \to 0$ limit of the form factors in the effective action can be found from the results quoted in the appendix, using $G_n(0,0,0) = -\frac{1}{2}\frac{\alpha}{\pi} g_n(0,0,0)$. From the definitions (4.4) we find:

$$\begin{aligned}
G_0(0) &= -\frac{1}{30}\frac{\alpha}{\pi} & G_1(0,0,0) &= -\frac{1}{144}\frac{\alpha}{\pi} & G_2(0,0,0) &= \frac{13}{360}\frac{\alpha}{\pi} \\
G_3(0,0,0) &= -\frac{1}{360}\frac{\alpha}{\pi} & G_4(0,0,0) &= \frac{1}{3360}\frac{\alpha}{\pi} & G_5(0,0,0) &= \frac{1}{315}\frac{\alpha}{\pi} \\
G_6(0,0,0) &= \frac{1}{2520}\frac{\alpha}{\pi} & G_7(0,0,0) &= \frac{1}{720}\frac{\alpha}{\pi} & G_8(0,0,0) &= \frac{2}{315}\frac{\alpha}{\pi} \\
G_9(0,0,0) &= -\frac{5}{1008}\frac{\alpha}{\pi}
\end{aligned} \tag{4.11}$$

and we confirm that $G_0(0)$, $G_{1,2,3}(0,0,0)$ correctly reproduce the coefficients d, a, b, c respectively in the Drummond-Hathrell action (1.2).

5. Numerical Analysis of the Form Factors

In order to get a better feeling for the behaviour of the form factors, we have evaluated them numerically. We present some of the results in graphical form in this section. The general features of the different form factors are quite similar, so here we just give the analysis for the form factors relevant for Ricci flat spaces, i.e. $g_3(0,x_2,x_3)$ and $g_9(0,x_2,x_3)$.

Collecting formulae from the appendix and substituting into eq.(4.4) we find

$$
\begin{aligned}
g_3(0,x_2,x_3) \quad = \quad &- \; F(0,x_2,x_3)\frac{1}{\Delta}\Big[4+\big(1+\frac{1}{4}(x_2+x_3)\big)(x_2+x_3)\Big] \\
&+ \; f(x_2)\frac{1}{\Delta}\Big[\frac{3x_2-x_3}{2x_2}+\frac{1}{4}(x_2-x_3)\Big(1-\frac{2(x_2+x_3)^2}{\Delta}\Big)\Big] \\
&+ \; f(x_3)\frac{1}{\Delta}\Big[\frac{3x_3-x_2}{2x_3}+\frac{1}{4}(x_3-x_2)\Big(1-\frac{2(x_2+x_3)^2}{\Delta}\Big)\Big] \\
&+ \; \frac{1}{\Delta}\Big[-1+\frac{1}{2}\Big(\frac{x_3}{x_2}+\frac{x_2}{x_3}+x_2+x_3\Big)\Big] \; + \; \frac{1}{12}\Big(\frac{1}{x_2}+\frac{1}{x_3}\Big) \; + \; \frac{1}{4}\Big(\frac{1}{x_2^2}+\frac{1}{x_3^2}\Big)
\end{aligned}
\tag{5.1}
$$

and

$$
\begin{aligned}
g_9(0,x_2,x_3) \quad = \quad &- \; F(0,x_2,x_3)\frac{1}{\Delta}\big[4+x_2+x_3\big] \\
&+ \; f(x_2)\Big[\frac{2}{\Delta^2}(x_2^2-x_3^2)-\frac{1}{2x_2^2}-\frac{2}{x_2^3}\Big] \\
&+ \; f(x_3)\Big[-\frac{2}{\Delta^2}(x_2^2-x_3^2)\Big] \; + \; f'(x_2)\Big(\frac{1}{x_2^2}+\frac{1}{2x_2}\Big) \; - \frac{2}{\Delta}+\frac{1}{3x_2^2}+\frac{2}{x_2^3}
\end{aligned}
\tag{5.2}
$$

The behaviour of these functions is illustrated in the following plots of $g_3(0,x_2,x_3)$ and $g_9(0,x_2,x_3)$. The values at the origin are $g_3(0,0,0)=\frac{1}{180}$ and $g_9(0,0,0)=\frac{5}{504}$, as already quoted in eq.(4.11). Notice that along the diagonals $x_2=x_3$, both functions tend asymptotically to zero. This is consistent with the vanishing of the BGZV form factors as $\Box\to\infty$ as can be seen from the definitions in eqs.(4.5) and (4.6). However, if one argument is set to zero, then the functions may tend to a finite limit. It is easy to check that this behaviour is entirely determined by the $h(x)$ and $h'(x)$ factors in eq.(4.4). These limits are $g_3(0,\infty,0)=g_3(0,0,\infty)=\frac{1}{60}$, $g_9(0,\infty,0)=0$ and $g_9(0,0,\infty)=\frac{1}{140}$ as can be seen in the figures below.

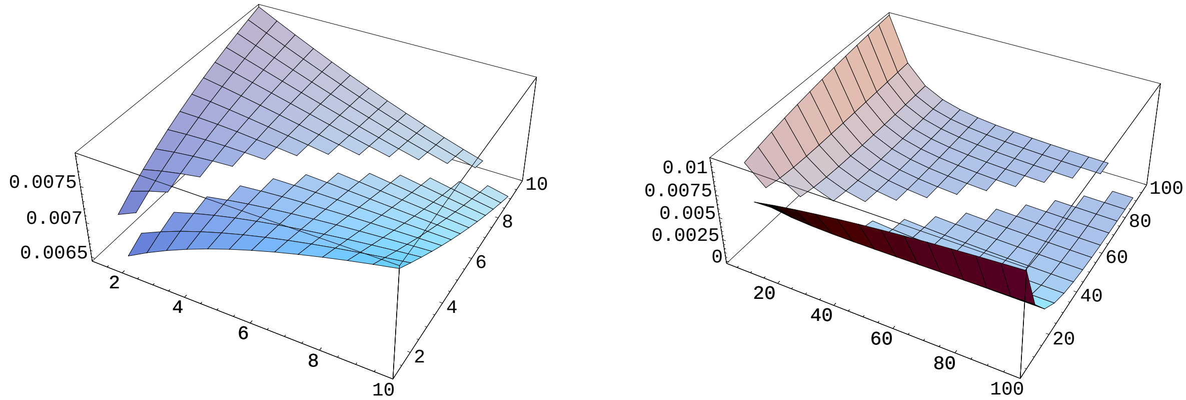

Fig.2 3D plots of $g_3(0, x_2, x_3)$ over different ranges.

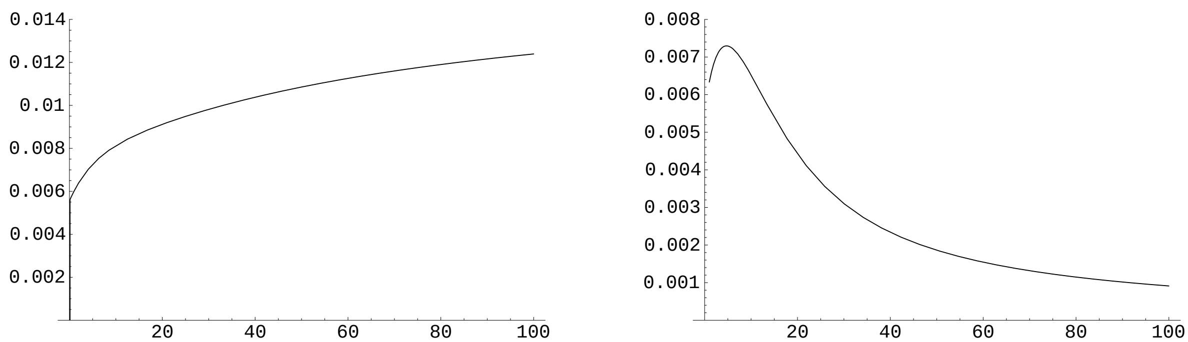

Fig.3 Graphs of $g_3(0, x_2, x_3)$ along the x_2 or x_3 axes (left) and the diagonal $x_2 = x_3$ (right).

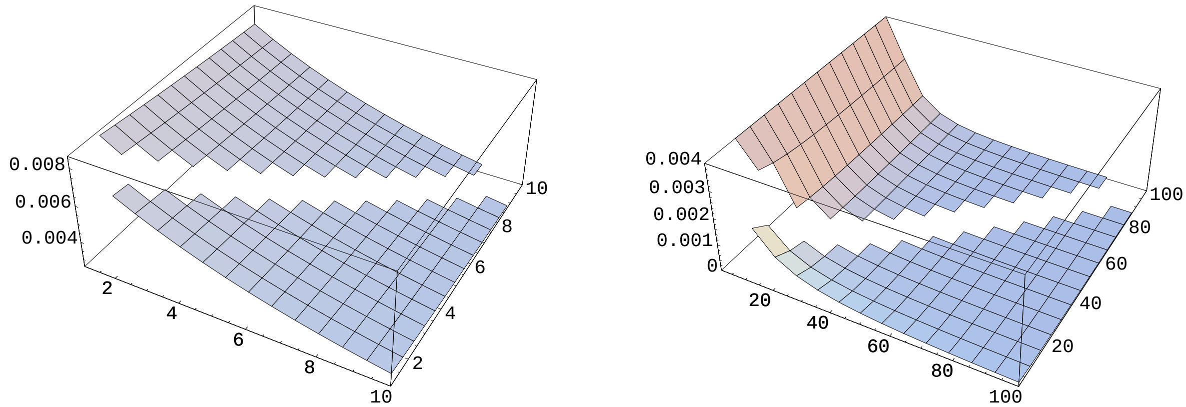

Fig.4 3D plots of $g_9(0, x_2, x_3)$ over different ranges.

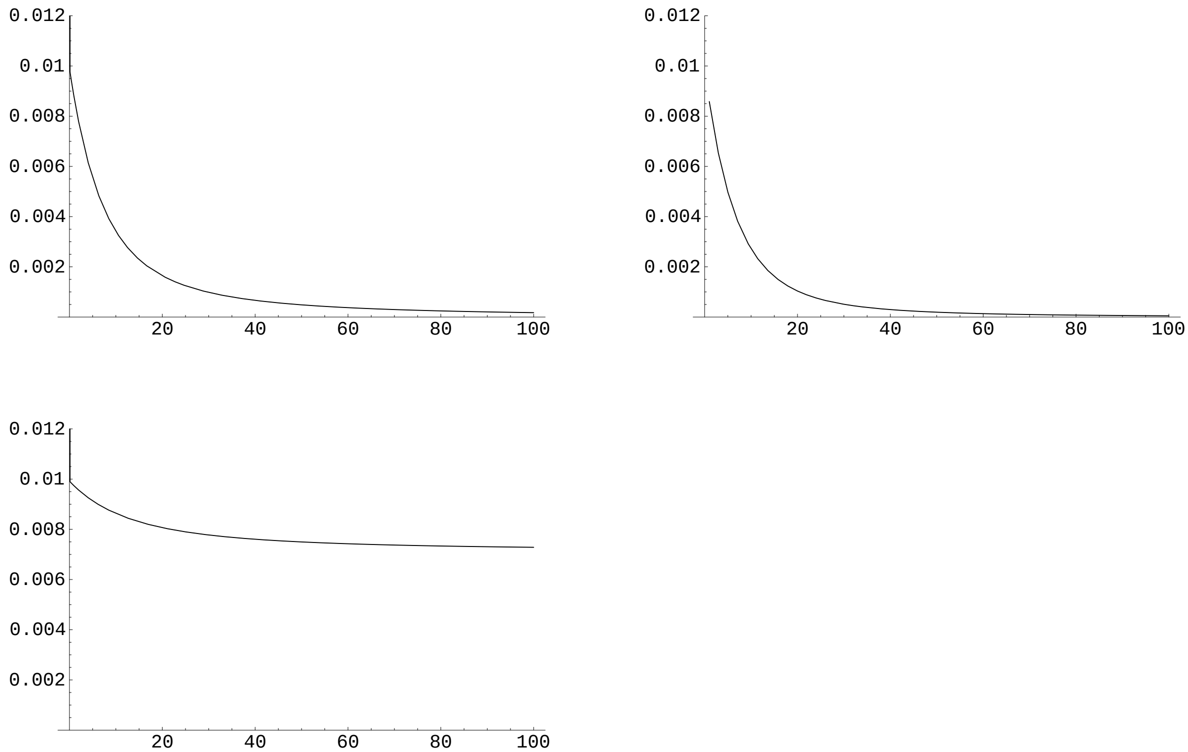

Fig.5 Graphs of $g_9(0, x_2, x_3)$ along the x_2 axis (top left), the diagonal $x_2 = x_3$ (top right) and the x_3 axis (lower).

This completes our numerical discussion of the form factors. We confirm that they are regular at $x_i = 0$ as required for their role in the local effective action. Their values at $x_i = 0$ match the well-known results for the Drummond-Hathrell effective action. They are smooth functions with well-understood asymptotic behaviour. The local effective action (4.2) for photon-gravity interactions induced by vacuum polarisation is therefore under complete algebraic and numerical control to all orders in the derivative expansion. The precise behaviour of the form factors may now be used to address a variety of physical applications, notably the important issue of dispersion for photon propagation in background gravitational fields [12].

Acknowledgements

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Appendix A.

In this appendix, we quote explicit formulae for the third order form factors appearing in the heat kernel in the limit discussed in section 4.

These are expressed in terms of the two basic integrals

$$f(x) = \int_0^1 d\alpha \; e^{-\alpha(1-\alpha)x} \tag{A.1}$$

and

$$F(0,x_2,x_3) = \frac{1}{x_2-x_3} \int_0^1 d\alpha \; \frac{1}{\alpha} \left(e^{-\alpha(1-\alpha)x_3} - e^{-\alpha(1-\alpha)x_2}\right) \tag{A.2}$$

We will at times use the further limit

$$\begin{aligned} F(0,x,0) &= \frac{1}{x} \int_0^1 d\alpha \; \frac{1}{\alpha}\left(1 - e^{-\alpha(1-\alpha)x}\right) \\ &= \frac{1}{2} - \frac{1}{24}x + \frac{1}{360}x^2 - \frac{1}{6720}x^3 + \ldots \end{aligned} \tag{A.3}$$

The final expressions also involve

$$F'(0,x_2,x_3) \equiv \frac{\partial}{\partial x_1} F(x_1,x_2,x_3)\Big|_{x_1=0} \tag{A.4}$$

which has a small x expansion

$$F'(0,x,0) = -\frac{1}{24} + \frac{1}{360}x - \frac{1}{6720}x^2 + \ldots \tag{A.5}$$

We also define $\Delta = (x_2 - x_3)^2$.

It is convenient to recall here the definition of $h(x)$ from eqs.(3.7),(4.8), viz.

$$\begin{aligned} h(x) &= -\frac{1}{6x} - \frac{1}{2x^2} + \left(\frac{1}{4x} + \frac{1}{2x^2}\right) f(x) \\ &= -\frac{1}{30} + \frac{1}{280}x + \ldots \end{aligned} \tag{A.6}$$

Beginning with the form factors which are not changed in the transformation to the local basis, we need the following contributions to $g_1(0,x_2,x_3)$. Note that these have been appropriately symmetrised.

$$F_1(0,x_2,x_3) = \frac{1}{3}F(0,x_2,x_3) \tag{A.7}$$

$$F_3(x_2,x_3,0) \;=\; F(0,x_2,x_3)\Big[-\frac{2}{\Delta}x_2x_3-\frac{2}{\Delta}(x_2+x_3)\Big] - \Big(f(x_2)-f(x_3)\Big)\frac{4}{\Delta^2}x_2x_3(x_2-x_3) \;+\; \frac{1}{\Delta}(x_2+x_3) \tag{A.8}$$

$$F_6(x_2,x_3,0) \;=\; F(0,x_2,x_3)\Big[-\frac{1}{6\Delta}(x_2^2+4x_2x_3+x_3^2)-\frac{1}{\Delta}(x_2+x_3)\Big] - \Big(f(x_2)-f(x_3)\Big)\frac{1}{4\Delta^2}(x_2-x_3)(x_2^2+6x_2x_3+x_3^2) \;+\; \frac{1}{2\Delta}(x_2+x_3) \tag{A.9}$$

$$\begin{aligned} F_7(0,x_2,x_3) \;=\; & F(0,x_2,x_3)\Big[\frac{1}{3\Delta^2}x_2x_3(x_2^2+4x_2x_3+x_3^2)+\frac{1}{3\Delta^2}(x_2+x_3)(x_2^2+34x_2x_3+x_3^2)\Big] \\ & +\Big(F(0,x_2,x_3)-\frac{1}{2}\Big)\frac{2}{\Delta^2}(x_2^2+10x_2x_3+x_3^2) \;-\; \frac{1}{\Delta^2}(x_2+x_3)x_2x_3 \\ & +\left[\frac{1}{2}\Big(\frac{f(x_2)-1}{x_2}\Big)\frac{1}{\Delta^3}x_2\big(x_2^4+44x_2^3x_3-10x_2^2x_3^2-36x_2x_3^3+x_3^4\big)+x_2\leftrightarrow x_3\right] \\ & +\left[\frac{1}{2}\Big(\frac{f(x_3)-1}{x_3}\Big)\frac{1}{\Delta^3}x_3\big(x_2^4-36x_3x_2^3-10x_3^2x_2^2+44x_3^3x_2+x_3^4\big)+x_2\leftrightarrow x_3\right] \end{aligned} \tag{A.10}$$

For $g_4(0,x_2,x_3)$ and $g_9(0,x_2,x_3)$, we require the following:

$$\begin{aligned} F_{14}(x_2,x_3,0) \;&=\; F_{14}(0,x_2,x_3) \\ &=\; F(0,x_2,x_3)\frac{2}{\Delta}(x_2+x_3) \;+\; \Big(F(0,x_2,x_3)-\frac{1}{2}\Big)\frac{8}{\Delta} \\ &\quad+\; \Big(f(x_2)-f(x_3)\Big)\frac{4}{\Delta^2}(x_2^2-x_3^2) \end{aligned} \tag{A.11}$$

$$\begin{aligned} F_{20}(0,x_2,x_3) \;=\; & F(0,x_2,x_3)\Big[-\frac{1}{3\Delta^2}(x_2+x_3)(x_2^2+4x_2x_3+x_3^2)-\frac{2}{3\Delta^2}(17x_2^2+38x_2x_3+17x_3^2)\Big] \\ & -\Big(F(0,x_2,x_3)-\frac{1}{2}\Big)\frac{24}{\Delta^2}(x_2+x_3) \;+\; \frac{1}{\Delta^2}(x_2+x_3)^2 \\ & -\left[f(x_2)\frac{2}{3\Delta^3}(x_2-x_3)(x_2+x_3)(x_2^2+4x_2x_3+x_3^2)+x_2\leftrightarrow x_3\right] \\ & -\left[\Big(\frac{f(x_2)-1}{x_2}\Big)\frac{1}{\Delta^3}(x_2-x_3)(21x_2^3+45x_2^2x_3+15x_2x_3^2-x_3^3)+x_2\leftrightarrow x_3\right] \end{aligned} \tag{A.12}$$

For $g_3(0,x_2,x_3)$ we need

$$F_3(0,x_2,x_3) \;=\; -\Big(f(x_2)-f(x_3)\Big)\frac{1}{2\Delta}(x_2-x_3) \tag{A.13}$$

while $g_7(0,x_2,x_3)$ involves

$$F_{17}(0,x_2,x_3) \;=\; -F(0,x_2,x_3)\frac{4}{\Delta} \;+\; \Big(f(x_2)+f(x_3)\Big)\frac{1}{\Delta} \tag{A.14}$$

We have explicitly checked that these form factors are indeed local by evaluating the limits $F_i(0,x,0)$ and verifying the cancellation of all the terms with inverse powers of x.

The remaining form factors involve the transformation (2.16) between the original non-local BGZV form factors and the new local ones. The first step is to identify the new form factor $\tilde{F}_0(0,x_2,x_3)$. We find this from any of the identities in eqs.(2.16). For example (remembering that we understand F_{18} to be symmetrised on x_2,x_3 and using the BGZV expression [13]),

$$\tilde{F}_0(0,x_2,x_3) \;=\; -\frac{1}{4}\lim_{x_1\to 0}\; x_1\; F_{18}(x_1,x_2,x_3) \tag{A.15}$$

All the identities produce the same result for $\tilde{F}_0(0,x_2,x_3)$, verifying the consistency of our method. We find

$$\begin{aligned}\tilde{F}_0(0,x_2,x_3) \;=\; & -\Big(F(0,x_2,x_3)-\frac{1}{2}\Big)\frac{4}{\Delta}\\ & +\Big(f(x_2)-1\Big)\frac{1}{2\Delta}\frac{1}{x_2}(3x_2-x_3) \;-\; \Big(f(x_3)-1\Big)\frac{1}{2\Delta}\frac{1}{x_3}(x_2-3x_3)\end{aligned} \tag{A.16}$$

The remaining form factors are then:

$$\begin{aligned}\tilde{F}_{18}(0,x_2,x_3) \;=\; & F(0,x_2,x_3)\frac{20}{\Delta^2}(x_2+x_3)^2\\ & +\Big(F(0,x_2,x_3)-\frac{1}{2}\Big)\frac{80}{\Delta^2}(x_2+x_3) \;+\; \frac{16}{\Delta}F'(0,x_2,x_3)\\ & +\Big[2\Big(\frac{f(x_2)-1}{x_2}\Big)\frac{1}{\Delta^3}(x_2-x_3)\big(17x_2^3+45x_2^2x_3+19x_2x_3^2-x_3^3\big) \;+x_2\leftrightarrow x_3\Big]\end{aligned} \tag{A.17}$$

$$\begin{aligned}\tilde{F}_{19}(0,x_2,x_3) \;=\; & -F(0,x_2,x_3)\frac{8}{\Delta^2}(x_2^2+3x_2x_3+x_3^2)\\ & -\Big(F(0,x_2,x_3)-\frac{1}{2}\Big)\frac{40}{\Delta^2}(x_2+x_3) \;-\; \frac{8}{\Delta}F'(0,x_2,x_3)\\ & +\Big[\Big(\frac{f(x_2)-1}{x_2}\Big)\Big(-\frac{8}{\Delta^3}x_2^2(x_2-x_3)(3x_2+7x_3)\\ & \qquad +\frac{12}{\Delta^4}x_2(x_2+x_3)\big(x_2^4-x_2^3x_3+6x_2^2x_3^2-4x_2x_3^2+x_3^4\big)\Big) \;+x_2\leftrightarrow x_3\Big]\end{aligned} \tag{A.18}$$

$$\begin{aligned}\tilde{F}_{21}(0,x_2,x_3) \;&=\; F(0,x_2,x_3)\frac{16}{\Delta^2}(x_2+x_3)(x_2+4x_3)\\
&+\;\Big(F(0,x_2,x_3)-\frac{1}{2}\Big)\frac{64}{\Delta^2}(x_2+4x_3)\;+\;\frac{32}{\Delta}F'(0,x_2,x_3)\\
&+\;\Big(\frac{f(x_2)-1}{x_2}\Big)\Big(\frac{8}{\Delta^3}(x_2-x_3)\big(9x_2^3+21x_2^2x_3+11x_2x_3^2-x_3^3\big)\;-\;\frac{48}{\Delta^2}x_2(x_2+x_3)\Big)\\
&+\;\Big(\frac{f(x_3)-1}{x_3}\Big)\Big(\frac{32}{\Delta^3}x_3(x_2-x_3)\big(x_2^2-6x_2x_3-5x_3^2\big)\;-\;\frac{48}{\Delta^2}x_3(x_2+x_3)\Big)
\end{aligned} \tag{A.19}$$

and finally

$$\begin{aligned}\tilde{F}_{8}(0,x_2,x_3) \;&=\; -\,F(0,x_2,x_3)\frac{40}{\Delta^2}x_2x_3(x_2+x_3)\\
&-\;\Big(F(0,x_2,x_3)-\frac{1}{2}\Big)\frac{160}{\Delta^2}x_2x_3\;-\;\frac{8}{\Delta}(x_2+x_3)F'(0,x_2,x_3)\\
&+\;\Bigg[\Big(\frac{f(x_2)-1}{x_2}\Big)\Big(-\frac{1}{\Delta^3}(x_2-x_3)\big(17x_2^4+86x_2^3x_3+72x_2^2x_3^2-22x_2x_3^3+7x_3^4\big)\\
&\qquad+\;\frac{6}{\Delta^4}(x_2+x_3)\big(3x_2^2+2x_2x_3-x_3^2\big)\Big)\;+x_2\leftrightarrow x_3\Bigg]
\end{aligned} \tag{A.20}$$

Again, using the results above, we have explicitly checked that the form factors $\tilde{F}_{0,8,18,19,21}$ all have regular small x limits, as required for the local operator basis. The required numerical cancellations are complicated and provide a very stringent test of the algebraic forms quoted.

Their remaining finite values when all of $x_{1,2,3}\to 0$ reproduce the coefficients in the Drummond-Hathrell action. To find these, we need the small x expansions of $f(x)$, $F(0,x,0)$ and $F'(0,x,0)$ given in eqs.(4.9),(A.3),(A.5). We find:

$$\tilde{F}_0(0,0,0)\;=\;\lim_{x\to 0}\Big[-\frac{4}{x^2}\Big(F(0,x,0)-\frac{1}{2}\Big)+\Big(\frac{f(x)-1}{x}\Big)\frac{3}{2x}+\frac{1}{12x}\Big]\;=\;\frac{1}{72} \tag{A.21}$$

and similarly

$$\begin{array}{llll}
F_1(0,0,0)=\dfrac{1}{6} & F_3(0,0,0)=\dfrac{1}{12} & F_6(0,0,0)=0 & F_7(0,0,0)=0\\
\tilde{F}_8(0,0,0)=-\dfrac{1}{180} & F_{14}(0,0,0)=\dfrac{1}{180} & F_{17}(0,0,0)=\dfrac{1}{180} & \tilde{F}_{18}(0,0,0)=\dfrac{1}{1260}\\
\tilde{F}_{19}(0,0,0)=-\dfrac{1}{1260} & F_{20}(0,0,0)=-\dfrac{1}{7560} & \tilde{F}_{21}(0,0,0)=\dfrac{1}{630} &
\end{array} \tag{A.20}$$

References

[1] I.T. Drummond and S. Hathrell, Phys. Rev. D22 (1980) 343.

[2] J.S. Schwinger, Phys. Rev. 82 (1951) 664.

[3] B.S. DeWitt, Dynamical Theory of Groups and Fields (Gordon and Breach, New York, 1965).

[4] S. Minakshisundaram, J. Indian Math. Soc. 17 (1953) 158.

[5] R.T. Seely, Proc. Symp. Pure Math. 10 (1967) 288.

[6] P.B. Gilkey, J. Diff. Geom. 10 (1975) 601.

[7] I.B. Khriplovich, Phys. Lett. B346 (1995) 251.

[8] G.M. Shore, Nucl. Phys. B460 (1996) 379.

[9] A.D. Dolgov and I.D. Novikov, Phys. Lett. B442 (1998) 82.

[10] G.W. Gibbons and C.A.R. Herdeiro, Phys. Rev. D63 (2001) 064006.

[11] G.M. Shore, Nucl. Phys. B605 (2001) 455.

[12] G.M. Shore, gr-qc/0203034.

[13] A.O. Barvinsky, Yu.V. Gusev, V.V. Zhytnikov and G.A. Vilkovisky, Print-93-0274 (Manitoba), 1993.

[14] A.O. Barvinsky and G.A. Vilkovisky, Nucl. Phys. B282 (1987) 163.

[15] A.O. Barvinsky and G.A. Vilkovisky, Nucl. Phys. B333 (1990) 471.

[16] A.O. Barvinsky and G.A. Vilkovisky, Nucl. Phys. B333 (1990) 512.

[17] I.G. Avramidi, Nucl. Phys. B355 (1991) 712.